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THESIS

A MATHEMATICAL ANALYSIS OF TACTICS
IN A RIVERINE AMBUSH

by

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13. ABSTRACT

This thesis examines different strategies for a patrol boat in a riverine ambush before and after the ambush.

Mathematical models employing concepts from games of strategy and statistical decision theory are used to study optimal tactics for the patrol boat before the ambush. A matrix game with the payoff a function of combat outcomes and combattant utility functions is used to study the optimal tactics for the boat and the ambushers. Using concepts of the statistical decision theory, various principles of choice are used to choose the appropriate decisions among all possible courses of action for the patrol boat.

Several combat models are used to investigate the patrol boat's tactics after the ambush has commenced. A deterministic Lanchester-type model with lethalities of fires that vary linearly with range is used to determine the casualty ratio between the two opponents. A stochastic model with constant attrition rates is used to calculate part of the probability distribution of the number of combattants alive at time t after the initiation of the ambush. Finally, a stochastic duel with displacement is used to determine the probability that one side would win in the ambush.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Ambush Military Tactics Games of Strategy Decision Theory Dynamic Combat Model Stochastic Model Stochastic Duel						

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I. INTRODUCTION AND BACKGROUND

The purpose of this thesis is to discuss some mathematical models of land combat that can be used to gain insight into tactics in a riverine ambush.

Lanchester-type models of combat and the theory of stochastic duels are used to determine the ambush outcomes at each stage in terms of the casualties sustained by the opposing forces. These models are used to estimate the consequences of different tactics for both the ambusher and ambushee. For each combination of tactics of the ambusher and ambushee, an entry for the payoff matrix of a general two-person game is generated. This game is then solved to determine the optimal tactic for each side.

In a land ambush, because of the surprise element in the ambush and because of the favorable terrain for the ambushers, defensive cover is initially minimal. As the engagement progresses, the ambushee seeks whatever cover is available and gradually improves his situation. The attachers, on the other hand, have a relative secure position that remains constant until the contest ends or until they choose to break off the engagement. The ambushees generally enter the contest by engaging in area fire, because of their lack of preparation for the immediate conflict. However as the battle unfolds, the defense maneuvers, attempts to locate the attackers, rushes the opponent's position if possible, and gradually switches from area fire to

aimed fire. The ambushers, on the other hand, engage in aimed fire throughout, although its net quality deteriorates with time [Ref. 1].

In a riverine ambush, the situation is somewhat different. First of all, due to the boat's superstructure, the ambushee does not need to seek cover elsewhere. He can stay right in his position and fire back at the ambushers. Secondly, since the ambushers have to choose the point of ambush close to the river, consequently they make themselves easy to be identified by the ambushees staying on the river, therefore the latter can enter the contest by engaging in aimed fire almost immediately from the start. Thirdly, due to the boat's maneuverability, the attackers do not enjoy the advantage of using aimed fire as effectively as in the case of a land ambush.

Nevertheless, in both types of ambushes, one is expected to get involved with the same basic factors of major importance which are so closely related to the general principles of guerrilla warfare that it is found necessary first to present some of these military thoughts before attempting to formulate the main problem. Needless to say, it is these principles that based upon them the commanders of both sides make their logical decisions at each stage of the fighting.

A. BASIC GUERRILLA TACTICS

Besides the political essence which is the vital characteristic of any guerrilla movement, the basic guerrilla

tactics are expressed operationally in the Communist's habitual refusal to accept combat unless vistory is certain. For the guerrilla fighters, one of the two insuring factors in the effort to try to be a winner in any engagement is accurate intelligence of both the enemy and the terrain. The second factor is the ability to concentrate secretly and vastly superior forces at the point of contact so that enemy units would be annihilated "one by one".

It is well remembered that, centuries ago, Sun Tzu already wrote: "Now war is based on deception. Move when it is advantageous, and create changes in the situation by dispersal and concentration of forces" [Ref. 2].

Today, Red China's ten principles are again the simple rules derived from the same basic thought but so effectively applied by the Communist Vietnam in its effort to try to take over the South that it is found worth-while to mention in this discussion some of them, especially the ones closely related to a riverine ambush.

First among the ten principles of the Chinese Reds is "...strike at scattered and isolated enemies, and later strike at the powerful enemies." Foremost in consistency and chronology was the North Vietnam's application of this axiom, which might be called the tactics of digestion without indigestion, a principle which was proportionate to the means at hand. The North Vietnam instructs the local Vietcong to attack the outposts, the patrolling boats before they try to attack the main bases.

The North Vietnam's forces also try to apply the fourth principle: "In every battle, concentrate absolutely superior forces.." This they usually do almost in every engagement, either small or big. An ambushed boat is also the subject of this axiom.

Consistently choosing its own conditions of battle, the North Vietnam adheres to the fifth principle: "Fight no unprepared engagements. Fight no engagements in which there is no assurance of victory." If they ambush a boat, they often choose to ambush from the right place and at the right time such as at the bends of the rivers and when the tide is low.

The North Vietnam's warriors are also subjected to the sixth principle which is "fear no sacrifice, fatigue" and to the seventh principle: "Strive to destroy the enemy while he is in movement" [Ref. 3].

B. COUNTER GUERRILLA TACTICS

Before entering into the discussion of the mechanics of killing the Communist Guerrillas, it is found necessary to mention is passing that, here again the political aspects such as follows are sine qua non:

1. Win the people. The bulk of this fight for the minds and hearts of the people must be a political fight, a fight waged by all means of the propaganda machine and most important of all, by the super examples of the true leaders from the highest levels to the lowest levels with

over-all emphasis on the high levels since only great leaders can produce great subordinates and not vice versa. Then the mechanics are:

2. Indoctrinate thoroughly the troops in the technique of political warfare and they must be familiar with their part in the war. The best potential counter-guerrilla force in any part of a country is a force from that part of that country.

Strange as it may be, tactics have always taken a rather back place to political, psychological, and intelligence factors in guerrilla wars that have been won [Ref. 4].

Military field tactics used against guerrillas are not unusual in any way. They are somewhat similar to the conventional operations with some variations.

Besides the all-out importance of the general theory of counter-guerrilla war such as to destroy the enemy lines of external support and to destroy the enemy's mobility, successful operations against guerrillas in small unit operations will often be the result of successful patrols and form an essential element of counter-insurgent warfare.

Routine patrolling seldom produces positive results. Because of the terrain, vegetation, and enemy tactics, modifications of normal techniques may be necessary. Patrol need to be all purpose: prepared to fight, ambush, pursue, and reconnoiter [Ref. 5]. These activities are applied to the infantry troops as well as to the riverine patrolling

crafts. However, for a patrol boat, it is more likely that she is subject to the ambushes than to anything else, therefore she should be well prepared in advance for such things as course, speed, special equipment, action if ambushed and method of attack.

II. FORMULATION OF THE PROBLEM

A. SCENARIO

Two homogeneous forces, a Blue force and a Red force were engaged in combat.

The Blue force was a patrol boat with many missions to carry, one of which was to patrol along an assigned river. This patrol had the purpose of trying to discourage the Red force's attempt to move men and supplies from the sanctuary areas, along or across the river, into the areas of operations. Another purpose of this kind of patrol was to protect the Blue force's supply and operations route either from the sea to the inland bases or between the inland bases themselves.

The Red force was the ambushing force whose purpose was to harass the Blue force's patrolling mission by trying to inflict to it as many casualties as possible.

The river in question was an imitation of the CUA-LON and BO-DE rivers in the NAM-CAN region in the southermost part of South Vietnam (Fig. 1). The river therefore was about three hundreds meters wide in the average. This meant that if the boat sailed close to one side of the river, she would be relatively safe from the ambushers rockets fired from the other side of the river although she was still well within the effective range of the rockets. The river was deep and navigable at all times even when the tide was low. It also had relative steep banks all along it even at the many bends where it changed directions

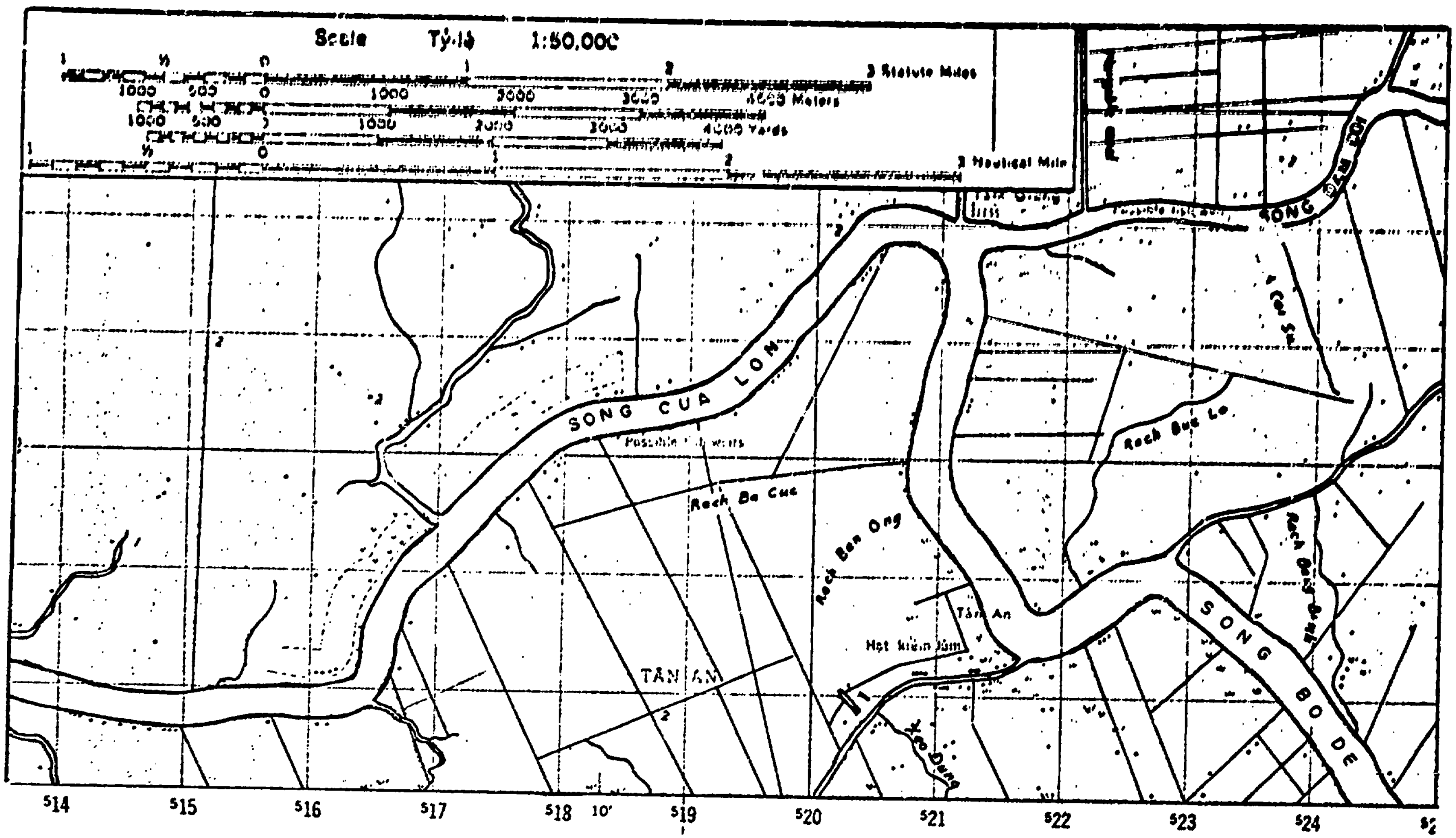


Figure 1. The CUA LON and BO DE Rivers.

sharply. These two factors suggested that the patrol boat could always sail close enough to either side of the river if she chose to do so.

According to past experiences that took place in the joint U.S.-VIENAMESE operations SEAFLOAT and SOLID ANCHOR during the years 1968-1970 in the NAM-CAN region, the Red force usually chose to set up the ambush at the bends of the river. There were several reasons which supported this decision.

First of all, at the bends of the river, since the current was relatively stronger than at the other places, the boat's maneuverability was greatly limited, therefore the boat's overall command to direct the counter-attack would become less effective.

Secondly, if the ambushers chose to ambush from the outside of the bend, they would enjoy the boat's minimum return fire power since right after the first firings, the boat would have to turn, hence to face her stern to the ambushers. It was logically assumed that onboard the boat, the guns were mostly distributed along her port and her star-board side.

Thirdly, at the bends of the river, the area was much broader in the outside than in the inside, consequently when the ambushers needed to disperse their force and ran away they would find the pursuing fire much less devastating.

Fourthly, since at the bends of the river, there usually were some small canals near-by, the ambushers would be able

to hide or to evade rather quickly and safely once they wanted to break contact with the boat.

B. GENERAL FORMULATION

The problem is divided in two main parts.

The first part consists of finding the optimal strategy before the ambush for the two players, namely player A or the Blue force and player B or the Red force.

The second part consists of finding the optimal strategy after the initial ambush for the same two players, namely, the patrol boat and the ambushers, assuming that both sides had used a certain set of pure strategies before the ambush. However, since the Red force has only two obvious strategies, i.e., either engage the enemy or break contact, this part deals mainly with the Blue force's strategies.

In all these two parts as well as in any military conflict between two opponents, the outcome or payoff depends on their decisions. Furthermore, a player does not know his opponent's strategy when he makes a decision. Therefore, each player must evaluate his opponent's capabilities, which in turn depend on the opponent's evaluation of the first side's capabilities, and these evaluations must be based on such essential elements as intelligence, fire power, survival probabilities and so on.

In a military conflict, the participants have opposing objectives - e.g., an attacker wishes to maximize the damage done to a defender's targets, while the defender wishes to minimize these same damage [Ref. 6].

Several game-theoretic models are used to gain insight into optimal tactics for both sides in a pre-formulated scenario.

Several different modelling methodologies are used to generate the payoff matrix entries in these games. In the first instance, deterministic Lanchester-type equations of warfare are used to determine the consequences of various alternative tactics for both sides. Other modelling methodologies using probabilistic approaches such as stochastic models are also used for the same purpose. Then a solution to the game would give instructions to each participant how best to choose from his available alternatives in order to best attain his objective.

Now, there are many ways to define and measure the combat effectiveness applied to the outcome or payoff of a military conflict.

According to Philip Hayward [Ref. 7], the only way of "measuring" the effectiveness of an organization, of whatever kind, is through the analysis of data on its performance under actual operating conditions. To obtain meaningful results from such analyses is often a difficult task, it is particularly difficult for an army combat unit, for the simple reason that the organization performs the tasks for which it was created only at rare intervals. If one wishes to measure or, more precisely, estimate P(S), the probability of success in combat operations, for a particular combat unit, enemy, environment, and mission, one must

find a number of cases filling the requirements and compute the frequency of success. Since historical records of combat are rarely compiled with this end of view, the research effort involved would be of formidable magnitude, particularly since the task would have to be repeated for each different situation. Furthermore, the results would apply only to combat units of the past, the effectiveness of new and untried organizations would remain unknown.

For these reasons, the problem of greater practical interest is the extent to which one can "predict" combat effectiveness on the basis of the empirical data, theory, and expert judgment available at the time.

Moreover, the most common measure of effectiveness applied to the outcome of a military engagement in guerrilla and counter-guerrilla operations has been the casualty ratio. A military commander today is presumed to be justified in sustaining heavy casualties to his own force if proportionately larger casualties are inflicted on the enemy, while a commander who suffers losses without inflicting greater harm on the adversary is judged a poor commander, regardless of the relative importance of the engagements in the overall conflict.

Since the purpose of this study is to provide military commanders with a realistic planning model, the casualty ratio will be accepted as a measure of effectiveness so that an optimum strategy can be chosen among the reasonable and available strategies without further justification.

Consequently, an optimal strategy for a player is defined as the strategy that gives the player the highest casualty ratio which is the ratio between the casualties suffered by his enemy and the casualties suffered by his own force.

III. BEFORE THE AMBUSH

A. A GAME OF STRATEGY

Before the ambush, the patrol boat's Captain can influence the outcome of the ambush by his behavior and he is interested in the outcome of the situation. The Commander of the ambushing forces can also influence the outcome by his behavior and he too is interested in the outcome of the situation. Thus the two players in this game would have to try to collect all available informations as accurate as possible about the terrain, about his own forces' capabilities as well as about all their possible courses of action.

The set of players is I = { Boat, Ambushers }

1. Strategies

Considering past combat experiences and all the informations mentioned above, it is assumed that each player has a finite number of strategies as follows (Fig. 2):

A_{boat} denotes the set of stategies of the patrol boat's Captain

$$A_{boat} = \{ A^1_{boat}, A^2_{boat}, A^3_{boat} \}$$

where:

A^1_{boat} = Sail the boat in the middle of the river

A^2_{boat} = Sail the boat close to the outside of the bend of the river

A^3_{boat} = Sail the boat close to the inside of the bend of the river

A_{ambush} denotes the set of strategies of the Commander of the ambushing forces

$$A_{ambush} = \{ A^1_{ambush}, A^2_{ambush} \}$$

where:

A^1_{ambush} = ambush from the outside of the bend of the river

A^2_{ambush} = ambush from the inside of the bend of the river

2. Elementary Outcomes

For this game, the following outcomes are mutually exclusive:

$e_1 \equiv$ the boat gets hit and the ambushers survive

$e_2 \equiv$ the boat gets hit and the ambushers are destroyed

$e_3 \equiv$ the boat is intact and the ambushers survive

$e_4 \equiv$ the boat is intact and the ambushers are destroyed

Note that the above outcomes are considered to be extremely general. One does not consider the special cases such as the payoffs of each side due to each player's respective strategies at each stage of the game after the ambush. Nor does one consider the strength as well as the fire-power of each side which may result in just one outcome, e.g., one side may be wiped out in just a few seconds after the ambush is started as one may see in the later parts of this study.

3. Outcome Functions

Let the probability of the outcomes e_i be p_i where $i = 1, 2, 3, 4$, one has the set of all mixed outcomes p:

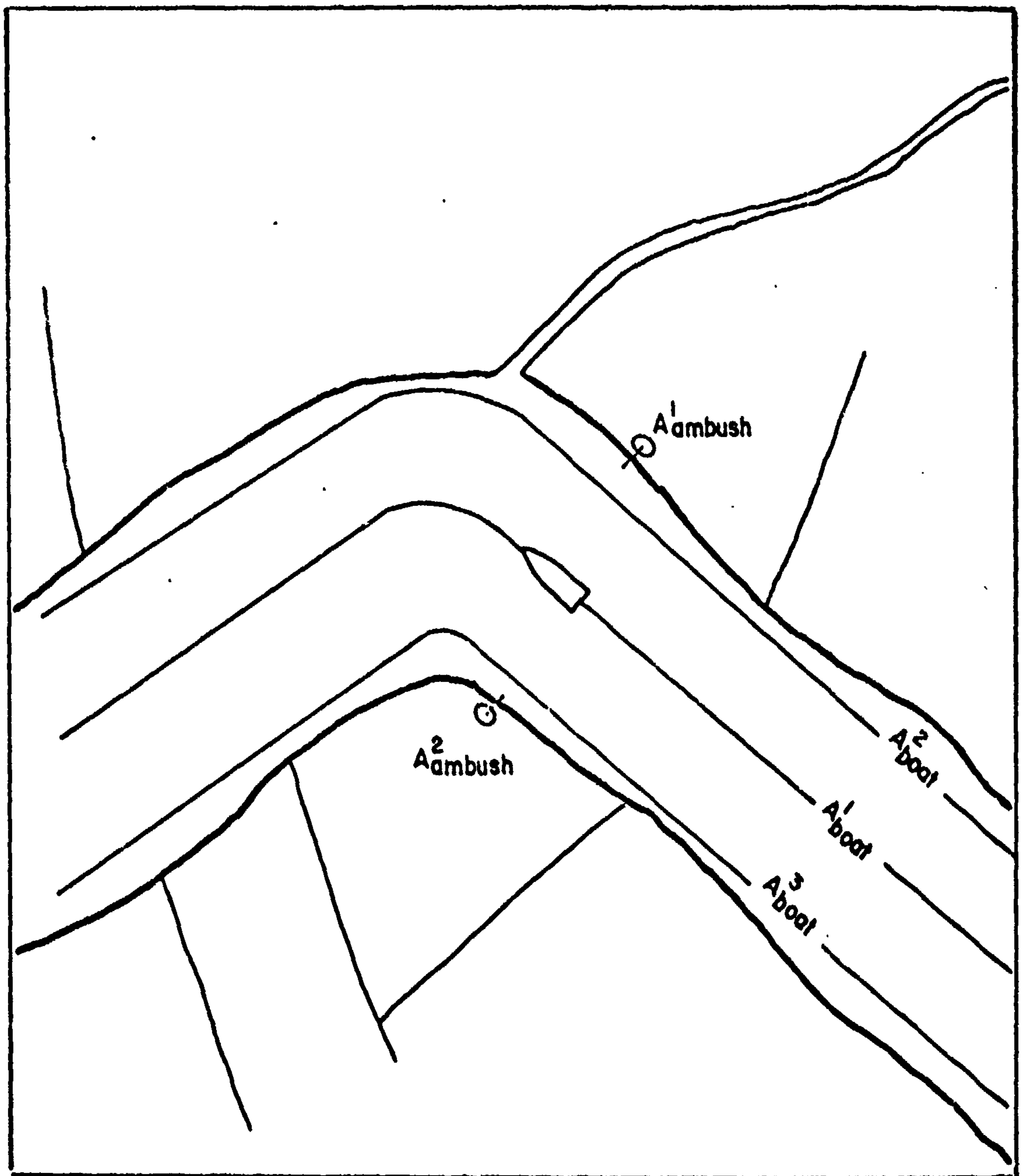


Figure 2. The Players' Strategies.

$p = (p_1, p_2, p_3, p_4)$

Based on the author's past combat experience and judgments, the probabilities of the following individual outcomes resulted from all possible combinations of courses of action of each player are assigned as follows:

a. Sail the boat in the middle of the river

(1) Get ambushed from the outside of the bend of the river

Probability (boat gets hit) = 0.8

Probability (ambushers survive) = 0.8

(2) Get ambushed from the inside of the bend of the river

Probability (boat gets hit) = 0.8

Probability (ambushers survive) = 0.4

b. Sail the boat close to the outside of the bend of the river

(1) Get ambushed from the outside of the bend of the river

Probability (boat gets hit) = 0.9

Probability (ambushers survive) = 0.7

(2) Get ambushed from the inside of the bend of the river

Probability (boat gets hit) = 0.1

Probability (ambushers survive) = 0.4

c. Sail the boat close to the inside of the bend of the river

(1) Get ambushed from the outside of the bend of the river

Probability (boat gets hit) = 0.1

Probability (ambushers survive) =0.9

(2) Get ambushed from the inside of the bend of the river

Probability (boat gets hit) = 0.9

Probability (ambushers survive) = 0.3

Therefore, one can get the following probabilities table for the outcome functions ρ

A_{boat}	A_{ambush}	p_1	p_2	p_3	p_4
A^1_{boat}	A^1_{ambush}	0.64	0.16	0.16	0.04
A^1_{boat}	A^2_{ambush}	0.32	0.48	0.08	0.12
A^2_{boat}	A^1_{ambush}	0.63	0.27	0.07	0.03
A^2_{boat}	A^2_{ambush}	0.05	0.05	0.45	0.45
A^3_{boat}	A^1_{ambush}	0.09	0.01	0.81	0.09
A^3_{boat}	A^2_{ambush}	0.27	0.63	0.03	0.07

4. Preference Relations and Utility Functions

A player's subjective probabilities are numerical representations of his beliefs and information. His utilities are numerical representations of his tastes and preferences [Ref. 8].

Now, as a matter of fact, the patrol boat's Captain would like his boat not to get hit and whether getting hit or not, he would like to wipe out the ambushing enemy.

Furthermore, since he still has other missions to carry besides his main mission of patrolling along the river, he would consider the fact that his boat gets hit and his enemy destroyed as equal preference to the fact that his boat is intact and his enemy can survive.

As for the ambushers, it is natural that they would prefer above all to hit the boat and to survive. Furthermore, since they are limited in numbers and their main mission is to harass the enemy as often as possible, they would prefer to survive after the ambush even at the price of not hitting the boat.

These feelings, expressed by the two players, define the following preference relations and utility functions:

The patrol boat's Captain

Preference relations:	$e_1 <$	$e_2 \sim$	$e_3 <$	e_4
Utility functions:	0	5	5	10

The ambushers

Preference relations:	$e_4 <$	$e_2 <$	$e_3 <$	e_1
Utility functions:	0	1	6	10

5. Payoff Functions

The outcome functions and the utility functions determine the following payoff functions table for all possible sets of pure strategies.

The payoff functions are obtained from the general formula

$$H_i\ (a_1, a_2, \ldots, a_n) = u_i\ [\rho(a_1, a_2, \ldots, a_n)]$$

where u_i is the utility function

$\rho(a_1, a_2, \ldots, a_n) = r$ is outcome function

$\underline{a} = (a_1, a_2, \ldots, a_n)$ is pure strategy vector

One can notice here that if:

$\underline{s} = (s_1, s_2, \ldots, s_n)$ is mixed strategy vector

then the outcome function for mixed strategy vector is

$$\rho(s_1, s_2, \ldots, s_n) = \sum_{a_1, a_2, \ldots a_n} \rho(a_1, a_2, \ldots, a_n) \, s_1(a_1) \ldots s_n(a_n)$$

and the payoff function for mixed strategy vector is

$$H_i(s_1, s_2, \ldots s_n) = u_i\,[\rho(s_1, s_2, \ldots, s_n)]$$

$$= \sum_{a_1, a_2, \ldots a_n} u_i[\rho(a_1, a_2, \ldots, a_n)] \, s_1(a_1) \ldots s_n(a_n)$$

$$= \sum_{a_1, \ldots a_n} H_i(a_1, \ldots, a_n) s_1(a_1) \ldots s_n(a_n)$$

It can be seen from the payoff function table that there is no optimal pure strategy for either of the two players. This is in fact a non cooperative two-person non-zero sum game.

A_{boat}	A_{ambush}	H_{boat}	H_{ambush}
1	1	0.80+0.80+0.40 = 2.00	6.40+0.16+0.96 = 7.52
1	2	2.40+0.40+1.20 = 3.00	3.20+0.48+0.48 = 4.16
2	1	1.35+0.35+0.30 = 2.00	6.30+0.27+0.42 = 6.99
2	2	0.25+2.25+4.50 = 7.00	0.50+0.05+2.70 = 3.25
3	1	0.05+4.05+0.90 = 5.00	0.90+0.01+4.86 = 5.77
3	2	3.15+0.15+0.70 = 4.00	2.70+0.63+0.18 = 3.51

6. Conclusion

Thus the game looks as follows:

$G = \{ I, C, A_i, H_i \}$

where $I = \{ \text{Boat, Ambushers} \} = C$

$A_{boat} = \{ A^1_{boat}, A^2_{boat}, A^3_{boat} \}$

$A_{ambush} = \{ A^1_{ambush}, A^2_{ambush} \}$

and the H_i are as specified in the payoff function table.

Due to dominance, this game can be described by the following payoff table:

		Red force 1	Red force 2
Blue force	2	(2, 6.99)	(7, 3.25)
	3	(5, 5.77)	(4, 3.51)

Let consider a pair of mixed strategies (x, 1-x) and (y, 1-y), i.e., the two players use their second and first strategy with probability x and y respectively. The expected payoffs are then

$$\bar{H}_{blue}(x,y) = 2xy + 7x(1-y) + 5(1-x)y + 4(1-x)(1-y)$$
$$= 4 + 3x + y(1-6x)$$

$$\bar{H}_{red}(x,y) = 6.99xy + 3.25x(1-y) + 5.77(1-x)y + 3.51(1-x)(1-y)$$
$$= 3.51 + 2.26y + x(0.26 - 0.48y)$$

From these expressions, when x = 1/6, the Blue force can secure an expected payoff of 4.5 for itself regardless

of what the opponent does. The Red force can in the same way make its expected payoff equal to 4.73 by choosing $y = 13/24$.

Note that if the Red force knows that the Blue force is an incarnate pessimist, bound to use a minimax strategy, it can put this knowledge to profitable use. From the expression

$$\bar{E}_{Red}\ (1/6,\ y) = 3.55 + 2.18y$$

the Red force will see that by choosing $y=1$, it can increase its expected payoff to 7.73. The Blue force may, however, argue in the same way and get higher expected payoff. Therefore it may be risky to depart from the minimax strategy.

Thus, since $x = 1/6$ and $y = 13/24$ for the minimax strategy, it is assumed that the Blue force would mostly use its third strategy, i.e., sail close to inside of the bend and that the Red force would use its first strategy, i.e., ambush from the outside of the bend. Therefore this set of pure strategies is assumed to be the starting point for the two forces after the initial ambush.

B. ALTERNATIVE SOLUTION: A STATISTICAL DECISION THEORY PROBLEM

A more sophisticated approach to the solution of the above problem is to model it as a statistical decision theory problem. In this approach, one would try to find how best one could make one's own decision among the available courses of action according to one's own principle.

Thus, in this case, one player would become a decision maker and his opponent's courses of action would be considered as the states of nature that the decision maker is goint to encounter.

The decision maker, without knowing the outcome of the engagement x as well as the state of nature w, must make a decision the consequences of which will depend on the outcome of the engagement as well as on the state of nature. Based on past combat experiences and/or judgments, it is assumed that there exists a probability distribution $F_w(x)$ on the space of the states of nature whose value is specified for each outcome x and that there exists a utility function u(r) on the set of the rewards r.

In this approach, the patrol boat's Captain will be treated as the decision maker. Tables 1, 2 and 3 will summarize all the data necessary for him based on which he could make various decisions according to his own principles which in turn may depend on the different military situations in which he is taking part.

Now, in this type of decision problem, it has become standard to specify for each reward $r \in R$ the negative of its utility, rather than its utility, and to call this number the loss. Hence, for each state of nature $w \in \Omega$ and each action $a \in A$, the loss $\ell(w,a)$ is defined by the equation:

$$\ell(w,a) = - U[\sigma(w,a)]$$

For any decision $d \in D$, the expected loss or risk $\rho(w,d)$ is specified by the equation:

$$\rho(w,d) = \int_{\Omega} \ell(w,a)\ dF_W\ (d(x)=a)$$

Table 4 and 5 give the loss function and all possible decisions available for the decision maker, namely, the patrol boat's Captain. Among all these decisions, only the decisions which are not inadmissible are admissible. A decision d within a class of decisions D is called inadmissible if there is another decision $d' \in D$ such that

$$\ell(w, d') \leq \ell(w, d) \quad \text{for all } w \in \Omega$$

and $$\ell(w, d') < \ell(w, d) \quad \text{for at least one } w \in \Omega$$

All admissible decisions functions for the problem of a riverine ambush are d_1, d_{19}, d_{21}, d_{27} and d_{81}.

<u>The principles of choice and the corresponding decision functions (Figs. 3 and 4)</u>:

1. <u>Principle of Insufficient Reason or Laplace</u>

According to this principle, for each $d \in D$, find the average loss $\bar{\ell}(d)=1/n \sum_{i=1}^{2} \ell(w_i, d)$ and $\bar{d}$ which minimize $\bar{\ell}(d)$. Therefore the decision that the boat's Captain would make is decision d_{21} which can be stated as: take action a_1 when the desired outcomes are x_1 and x_3 and take action a_3 when the desired outcomes are x_2 and x_4.

2. <u>Minimax Principle or Von Neumann and Morgenstern Principle</u>

According to this principle, for each $d \in D$, find $\sup_{w \in \Omega} \ell(w,d)$ and then find $d^\circ \in D$ which minimizes this max.

Outcome function $\sigma(w,d)=r$	Boat's action		
	a_1 Sail close to the inside of the bend	a_2 Sail in the middle of the river	a_3 Sail close to the outside of the bend
w_1 Ambushers from the inside of the bend	r_4	r_3	r_1
w_2 Ambushers from the outside of the bend	r_2	r_6	r_5

Table 1. Outcome Function.

Probability Distribution $F_W(x)$	Outcome of the engagement			
	x_1 Boat gets hit & ambushers survive	x_2 Boat gets hit & ambushers destroyed	x_3 Boat's intact & ambushers survive	x_4 Boat's intact & ambushers destroyed
w_1 Ambushers from the inside of the bend	0.1	0.4	0.2	0.3
w_2 Ambushers from the outside of the bend	0.2	0.2	0.35	0.25

Table 2. Probability Distribution.

Result r	Utility (u(r))
r_1	0.5
r_2	0.2
r_3	-0.1
r_4	-0.2
r_5	-0.3
r_6	-0.5
Zero utility: status quo	
Range of utility = 1.0	

Table 3. Unility Function.

Loss function $\ell(w,a) = -10u(\sigma(w,a))+5$	Action		
	a_1	a_2	a_3
w_1	7	6	0
w_2	2	10	8

Table 4. Loss Function.

Decision Function $d(x)=a$					Action Probability $P_w(x-d(x)=a)$						Risk $\rho(w,d)=\sum_{i=1}^{3} \ell(w_i,a_i)P_w(d(x)=a)$	
					w_1			w_2				
d	x_1	x_2	x_3	x_4	a_1	a_2	a_3	a_1	a_2	a_3	w_1	w_2
d_1	a_1	a_1	a_1	a_1	1	0	0	1	0	0	7	2
d_2	a_1	a_1	a_1	a_2	0.7	0.3	0	0.75	0.25	0	6.7	4.0
d_3	a_1	a_1	a_1	a_3	0.7	0	0.3	0.75	0	0.25	4.9	3.5
d_4	a_1	a_1	a_2	a_1	0.8	0.2	0	0.65	0.35	0	7.5	4.8
d_5	a_1	a_1	a_2	a_2	0.5	0.5	0	0.4	0.6	0	6.5	6.8
d_6	a_1	a_1	a_2	a_3	0.5	0.2	0.3	0.4	0.35	0.25	4.7	6.3
d_7	a_1	a_1	a_3	a_1	0.8	0	0.2	0.65	0	0.35	5.6	4.1
d_8	a_1	a_1	a_3	a_2	0.5	0.3	0.2	0.4	0.25	0.35	5.3	6.1
d_9	a_1	a_1	a_3	a_3	0.5	0	0.5	0.4	0	0.6	3.5	5.6
d_{10}	a_1	a_2	a_1	a_1	0.6	0.4	0	0.8	0.2	0	7.1	3.6
d_{11}	a_1	a_2	a_1	a_2	0.3	0.7	0	0.55	0.45	0	5.3	5.6
d_{12}	a_1	a_2	a_1	a_3	0.3	0.4	0.3	0.55	0.2	0.25	4.5	5.1
d_{13}	a_1	a_2	a_2	a_1	0.4	0.6	0	0.45	0.55	0	6.4	6.4
d_{14}	a_1	a_2	a_2	a_2	0.1	0.9	0	0.2	0.8	0	6.1	8.4
d_{15}	a_1	a_2	a_2	a_3	0.1	0.6	0.3	0.2	0.55	0.25	4.3	7.9
d_{16}	a_1	a_2	a_3	a_1	0.4	0.4	0.2	0.45	0.2	0.35	5.2	5.7
d_{17}	a_1	a_2	a_3	a_2	0.1	0.7	0.2	0.2	0.45	0.35	4.9	7.7
d_{18}	a_1	a_2	a_3	a_3	0.1	0.4	0.5	0.2	0.2	0.6	3.1	7.2
d_{19}	a_1	a_3	a_1	a_1	0.6	0	0.4	0.8	0	0.2	4.2	3.2
d_{20}	a_1	a_3	a_1	a_2	0.3	0.3	0.4	0.55	0.25	0.2	3.9	5.2
d_{21}	a_1	a_3	a_1	a_3	0.3	0	0.7	0.55	0	0.45	2.1	4.7
d_{22}	a_1	a_3	a_2	a_1	0.4	0.2	0.4	0.45	0.35	0.2	3.9	6.0
d_{23}	a_1	a_3	a_2	a_2	0.1	0.5	0.4	0.2	0.6	0.2	3.7	8.0
d_{24}	a_1	a_3	a_2	a_3	0.1	0.2	0.7	0.2	0.35	0.45	1.9	7.5
d_{25}	a_1	a_3	a_3	a_1	0.4	0	0.6	0.45	0	0.55	2.8	5.3
d_{26}	a_1	a_3	a_3	a_2	0.1	0.3	0.6	0.2	0.25	0.55	2.5	7.3
d_{27}	a_1	a_3	a_3	a_3	0.1	0	0.9	0.2	0	0.8	0.7	6.8
d_{28}	a_2	a_1	a_1	a_1	0.9	0.1	0	0.8	0.2	0	6.9	3.6
d_{29}	a_2	a_1	a_1	a_2	0.6	0.4	0	0.55	0.45	0	6.6	5.6

d_{30}	a_2	a_1	a_1	a_3	0.6	0.1	0.3	0.55	0.2	0.25	4.8	5.1
d_{31}	a_2	a_1	a_2	a_1	0.7	0.3	0	0.45	0.55	0	6.7	6.4
d_{32}	a_2	a_1	a_2	a_2	0.4	0.6	0	0.2	0.8	0	6.4	8.4
d_{33}	a_2	a_1	a_2	a_3	0.4	0.3	0.3	0.2	0.55	0.25	4.6	7.9
d_{34}	a_2	a_1	a_3	a_1	0.7	0.1	0.2	0.45	0.2	0.35	5.5	5.7
d_{35}	a_2	a_1	a_3	a_2	0.4	0.4	0.2	0.2	0.45	0.35	5.2	7.7
d_{36}	a_2	a_1	a_3	a_3	0.4	0.1	0.5	0.2	0.2	0.6	3.4	7.2
d_{37}	a_2	a_2	a_1	a_1	0.5	0.5	0	0.6	0.4	0	6.5	5.2
d_{38}	a_2	a_2	a_1	a_2	0.2	0.8	0	0.35	0.65	0	6.2	7.2
d_{39}	a_2	a_2	a_1	a_3	0.2	0.5	0.3	0.35	0.4	0.25	4.4	6.7
d_{40}	a_2	a_2	a_2	a_1	0.3	0.7	0	0.25	0.75	0	6.3	8.0
d_{41}	a_2	a_2	a_2	a_2	0	1	0	0	1	0	6	10
d_{42}	a_2	a_2	a_2	a_3	0	0.7	0.3	0	0.75	0.25	4.2	9.5
d_{43}	a_2	a_2	a_3	a_1	0.3	0.5	0.2	0.25	0.4	0.35	5.1	7.3
d_{44}	a_2	a_2	a_3	a_2	0	0.8	0.2	0	0.65	0.35	4.8	9.3
d_{45}	a_2	a_2	a_3	a_3	0	0.5	0.5	0	0.4	0.6	3.0	8.8
d_{46}	a_2	a_3	a_1	a_1	0.5	0.1	0.4	0.6	0.2	0.2	4.1	5.8
d_{47}	a_2	a_3	a_1	a_2	0.2	0.4	0.4	0.35	0.45	0.2	3.8	6.8
d_{48}	a_2	a_3	a_1	a_3	0.2	0.1	0.7	0.35	0.2	0.45	2.0	6.9
d_{49}	a_2	a_3	a_2	a_1	0.3	0.3	0.4	0.25	0.55	0.2	1.9	7.6
d_{50}	a_2	a_3	a_2	a_2	0	0.6	0.4	0	0.8	0.2	3.6	9.6
d_{51}	a_2	a_3	a_2	a_3	0	0.3	0.7	0	0.55	0.45	1.8	9.1
d_{52}	a_2	a_3	a_3	a_1	0.3	0.1	0.6	0.25	0.2	0.55	2.7	6.9
d_{53}	a_2	a_3	a_3	a_2	0	0.4	0.6	0	0.45	0.55	2.4	8.9
d_{54}	a_2	a_3	a_3	a_3	0	0.1	0.9	0	0.2	0.8	0.6	8.4
d_{55}	a_3	a_1	a_1	a_1	0.9	0	0.1	0.8	0	0.2	6.3	3.2
d_{56}	a_3	a_1	a_1	a_2	0.6	0.3	0.1	0.55	0.25	0.2	6.0	5.2
d_{57}	a_3	a_1	a_1	a_3	0.6	0	0.4	0.55	0	0.45	4.2	4.7
d_{58}	a_3	a_1	a_2	a_1	0.7	0.2	0.1	0.45	0.35	0.2	6.1	6.0
d_{59}	a_3	a_1	a_2	a_2	0.4	0.5	0.1	0.2	0.6	0.2	5.8	8.0
d_{60}	a_3	a_1	a_2	a_3	0.4	0.2	0.4	0.2	0.35	0.45	4.0	7.5
d_{61}	a_3	a_1	a_3	a_1	0.7	0	0.3	0.45	0	0.55	4.9	5.3
d_{62}	a_3	a_1	a_3	a_2	0.4	0.3	0.3	0.2	0.25	0.55	4.6	7.3
d_{63}	a_3	a_1	a_3	a_3	0.4	0	0.6	0.2	0	0.8	2.8	6.8
d_{64}	a_3	a_2	a_1	a_1	0.5	0.4	0.1	0.6	0.2	0.2	5.9	4.8
d_{65}	a_3	a_2	a_1	a_2	0.2	0.7	0.1	0.35	0.45	0.2	5.6	6.8

d_{66}	a_3	a_2	a_1	a_3	0.2	0.4	0.4	0.35	0.2	0.2	3.8	4.3
d_{67}	a_3	a_2	a_2	a_1	0.3	0.6	0.1	0.25	0.55	0.2	5.7	7:6
d_{68}	a_3	a_2	a_2	a_2	0	0.9	0.1	0	0.8	0.2	5.4	9.6
d_{69}	a_3	a_2	a_2	a_3	0	0.6	0.4	0	0.55	0.45	3.6	9.1
d_{70}	a_3	a_2	a_3	a_1	0.3	0.4	0.3	0.25	0.2	0.55	4.5	6.9
d_{71}	a_3	a_2	a_3	a_2	0	0.7	0.3	0	0.45	0.55	4.2	8.9
d_{72}	a_3	a_2	a_3	a_3	0	0.4	0.6	0	0.2	0.8	2.4	8.4
d_{73}	a_3	a_3	a_1	a_1	0.5	0	0.5	0.6	0	0.4	3.5	4.4
d_{74}	a_3	a_3	a_1	a_2	0.2	0.3	0.5	0.35	0.25	0.4	3.8	6.4
d_{75}	a_3	a_3	a_1	a_3	0.2	0	0.8	0.35	0	0.65	1.4	5.9
d_{76}	a_3	a_3	a_2	a_1	0.3	0.2	0.5	0.25	0.35	0.4	3.3	7.2
d_{77}	a_3	a_3	a_2	a_2	0	0.5	0.5	0	0.6	0.4	3.0	9.2
d_{78}	a_3	a_3	a_2	a_3	0	0.2	0.8	0	0.35	0.65	1.2	8.7
d_{79}	a_3	a_3	a_3	a_1	0.3	0	0.7	0.25	0	0.75	2.1	6.5
d_{80}	a_3	a_3	a_3	a_2	0	0.3	0.7	0	0.25	0.75	1.8	8.5
d_{81}	a_3	a_3	a_3	a_3	0	0	1	0	0	1	0	8.0

Table 5. Risk Function.

The minimax loss $\bar{v}$ is

$$\bar{v} = \inf_{d \varepsilon D} \text{mp}_{w \varepsilon \Omega} \ell(w, d) = \sup_{w \varepsilon \Omega} \ell(w, d^{\circ})$$

Thus the decision the boat's Captain would make if he follows this principle is decision $d_{minimax} = (1/3\ d_{21}, 2/3\ d_{19})$ which can be stated as: One out of three times, use decision d_{21} and two out of three times, use decision d_{19} which suggests to take action a_1 when the desired outcomes are x_1, x_2 and x_4 and to take action a_3 when the desired outcome is x_3.

3. Principle of Minimax Regret or Savage

According to this principle, instead of working with the loss $\ell(w, a)$, compute the regret $g(w, a) = \ell(w, a) - \inf_{a \varepsilon A} \ell(w, a)$, then apply Minimax Principle to (Ω, A, g).

Thus, if following this principle, the boat's Captain would take decision $d_{Savage} = (4/27\ d_{19}, 23/27\ d_{21})$.

4. Principle of Pessimism-Optimism or Hurwicz

According to this principle, choose a number $0 \leqslant \alpha \leqslant 1$ such that

$$\alpha \sup_{w \varepsilon \Omega} \ell(w,d) + (1-\alpha) \inf_{w \varepsilon \Omega} \ell(w,d) = \ell_{\alpha}(d)$$

then find d which minimizes $\ell_{\alpha}(d)$. α is called degree of pessimism. Thus if $\alpha = 0.8$, the boat's Captain would make decision $d_{Hurwicz}$ which happens to be $d_{minimax}$ in this case. If $\alpha = 0.2$, he would make decision d_{81} which suggests to take action a_3 all the time.

5. Bayes Principle

Let P or prior distribution be a probability distribution on Ω then compute the Bayes risk $\rho(P,d)$

$$\rho(P,d) = E_P\{\ell(w,d)\} = \int_\Omega \ell(w,d)\,dP(w)$$

then find d* which minimizes $\rho(P,d)$. Thus according to this principle, there are an infinite number of decisions which depend on the prior distribution and which form a Bayes Risk functional $\rho^*(P)$. Suppose P = (1/5, 4/5) then this principle suggests to use decision d_1 which states that the boat's Captain should always take action a_1 (Fig. 4).

6. Conclusion

It can be seen from Fig. 4 that when the prior P has the value for the probability of w_2 greater than 73/127, it is suggested that the boat's Captain should always take action a_3 which is to sail his boat close to the outside of the bend. The reason is that the patrol boat's Captain is assumed to prefer the outcome of the engagement to be x_4 which is the situation where the boat is intact and the ambushers are destroyed.

In any case, for the boat's Captain, only action a_2 and a_3 are worth consideration. Under no circumstances should he take action a_1 which is to sail his boat in the middle of the river.

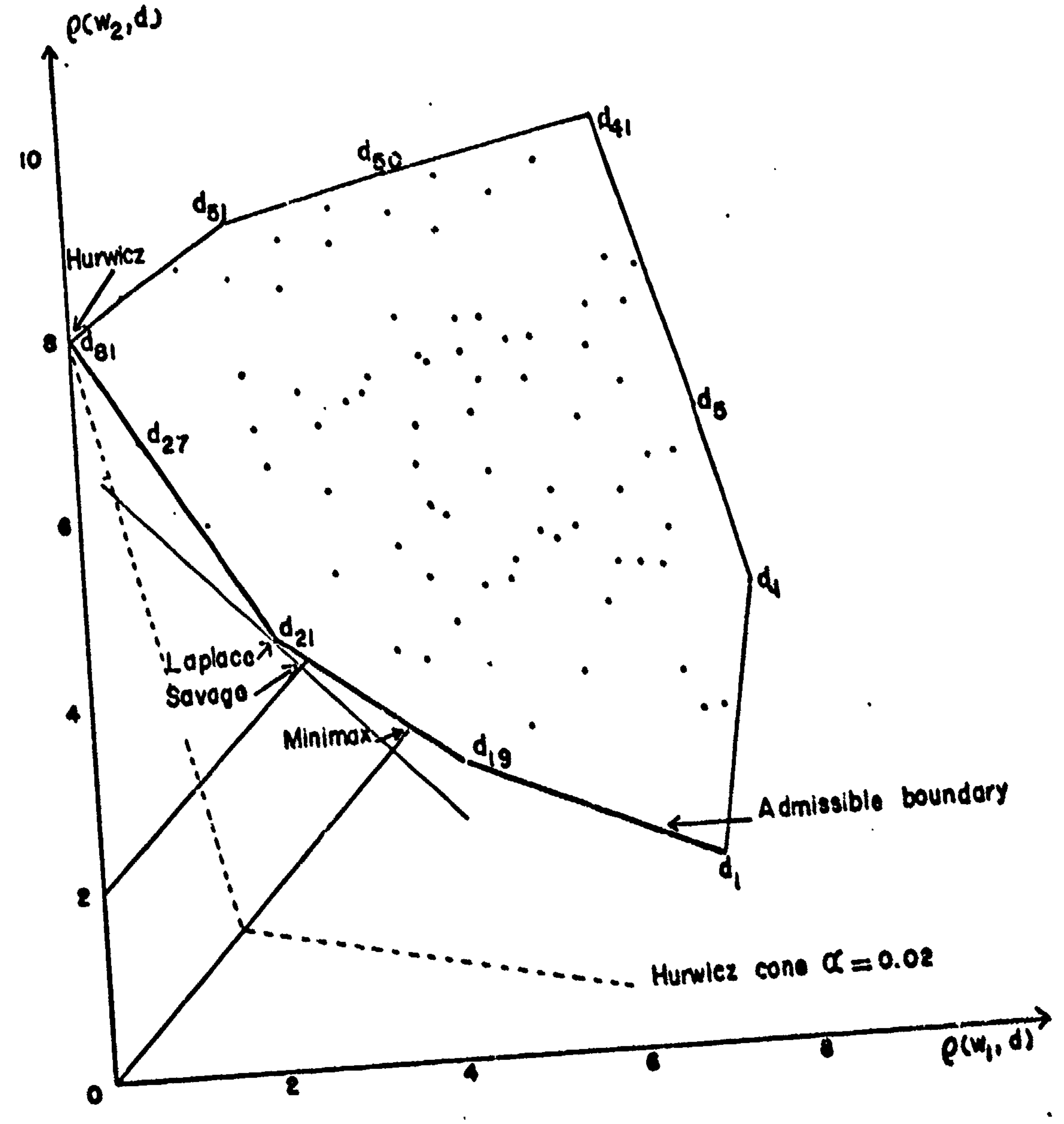


Figure 3. Risk Set in D-Space.

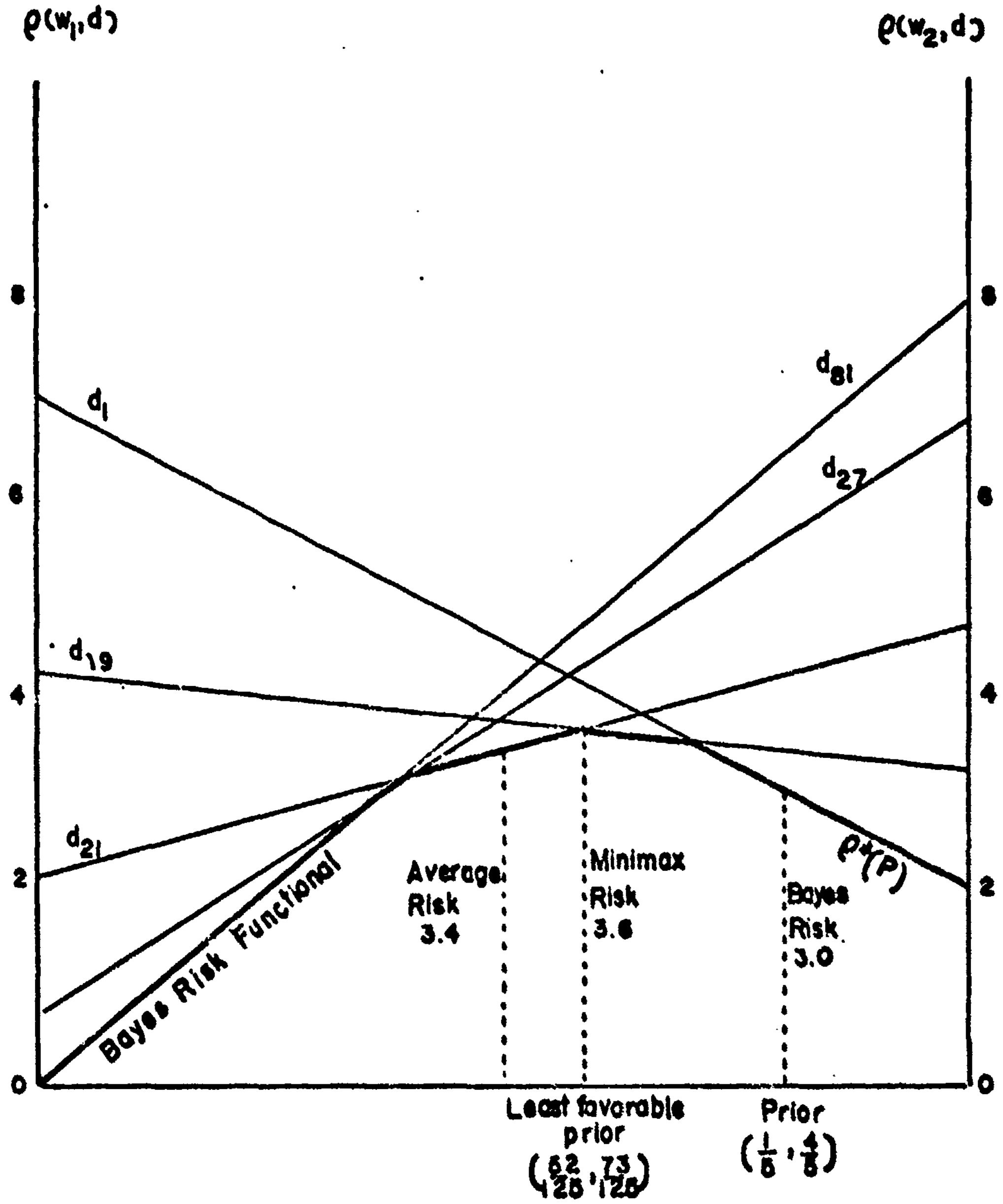


Figure 4. Bayes Risk Functional.

IV. AFTER THE INITIAL AMBUSH: SOME COMBAT MODELS

A. GENERAL NOTIONS

Military experience has long suggested that the outcome of an engagement - as opposed to a war - is dependent on the interaction between weapon characteristics and the tactics employed [Ref. 9].

Several different modelling techniques are used to forecast combat outcomes when different tactics are used. These techniques combine the weapon system performance with the tactics. Specifically, this study considers deterministic and stochastic models applied in insurgent and counter-insurgent warfare.

Deterministic Lanchester-type models of warfare are commonly considered to deal with the losses on opposing sides when large numbers of combatants are involved and when various assumptions about the loss rates are made. Although not mathematically correct, solutions of the deterministic equations are frequently interpreted as expected numbers of surviving combattants [Ref. 10].

However, Snow (1948) [Ref. 11] indicated that the expected value solutions imply underlying probability distributions and suggested a stochastic analysis of Lanchester's equations. Snow and Morse and Kimball [Ref. 12] have shown that in simple cases at least, the difference-differential equations applicable to the probabilistic treatment give a good approximation to the Lanchester's equations, especially for small t.

Another interesting approach to the stochastic models is the theory of stochastic duels which is concerned with the microscopic features of combat such as individual kill probabilities, time between rounds fired, ammunition limitations, cover, concealment, surprise, mobility, and so forth. This is a sharp distinction to Lanchester's theory which aggregates all these effects [Ref. 13].

Now, for the patrol boat, after the initial ambush, there are several available courses of action or tactics to be employed. In the patrol boat Captain's planning horizon, these tactics depend on how he judges the general situation, i.e., how he estimates his enemy's strength such as force size, type of weapons and fire-power. They also depend on the various mathematical models of combat used to find the outcome of the fighting.

Thus, depending on different combat situations and different combat models which in turn depend on the assumptions made according to the models themselves and to the tactics being used, one can have various solutions to the problem at hand.

B. FIRST SOLUTION: A DYNAMIC COMBAT MODEL

One of the extensions in the development of the Lanchester theory of combat is a model of dynamic combat which incorporates the effects of mobility and range-varying attrition rates on the outcome of an engagement.

1. The Strategies

After the initial ambush, although the ambushers or the Red force have already lost their initial advantage, they are nevertheless assumed to go on with the fighting in the hope of inflicting more damage to the boat until either the number of their casualties approaches the order of n percent or the boat breaks contact.

As for the ambushees or the Blue force, from their position close to the inside of the bend of the river, the patrol boat's Captain, according to past combat experiences and/or judgments of the military experts, is assumed to have three options:

a. Engage the ambushers with small arms by turning around and keeping the boat close to the inside of the bend of the river (course A - Fig. 5) as many times as necessary until either the enemy breaks contact or the number of his casualties approaches the order of m percent. If the enemy breaks contact, then the patrol boat would use artillery for p minutes. The number of tube of artillery available onboard the patrol boat at this moment is g.

b. Engage the ambushers with small arms in an excessive way by turning around and going straight to land right in front of the enemy, thus causing the enemy to abandon his hiding and break contact (course B - Fig. 5). The patrol boat then uses artillery for p minutes. The number of tube of artillery available aboard the patrol boat at this moment is g.

c. Turn left and try to break contact by going straight ahead then use artillery for p minutes once out of the effective firing range of the enemy. The number of tube of artillery available onboard the patrol boat is again g.

2. The Model

Bonder has developed the extensions of Lanchester equations to investigate the effects of mobility and range dependencies of weapon systems. He has done this by formulating a model which considers mobility and the influence of range on the attrition-rate coefficients. The attrition equations of this model are given by:

$$\frac{dx}{dt} = -a(r)y \tag{1}$$

$$\frac{dy}{dt} = -b(r)x \tag{2}$$

where:

x = number of Blue force survivors at time t

y = number of Red force survivors at time t

$a(r)$ = Red force weapon attrition rates or kill rates or the rate at which a single y unit destroys a single x unit

$b(r)$ = Blue force weapon attrition rates or kill rates or the rate at which a single x unit destroys a single y unit.

By the chain rule

$$\frac{dx}{dt} = \frac{dx}{dr} \cdot \frac{dr}{dt} = v \frac{dx}{dr}$$

where v is the speed of the Blue force and r is the range from static Red force to mobile Blue force, equations (1) and (2) become

$$v \frac{dx}{dr} = -a(r)y \tag{3}$$

$$v \frac{dy}{dt} = -b(r)x \tag{4}$$

when v = v(r) and the ratio of attrition rate constant

$$a(r) = k_a\, g(r)$$

$$b(r) = k_b\, g(r)$$

equations (3) and (4) have the following solution [Ref. 14] after being transformed into second order differential equations with initial conditions $x(r = r_o) = x_o$ and $y(r = r_o) = y_o$:

$$x(r) = x_o \cosh \Theta(r) - \sqrt{\frac{k_a}{k_b}}\, y_o \sinh \Theta(r) \tag{5}$$

$$y(r) = y_o \cosh \Theta(r) - \sqrt{\frac{k_b}{k_a}}\, x_o \sinh \Theta(r) \tag{6}$$

where $\Theta(r) = \sqrt{k_a k_b} \int_{r_o}^{r} \frac{g(r)}{v(r)}\, dr$

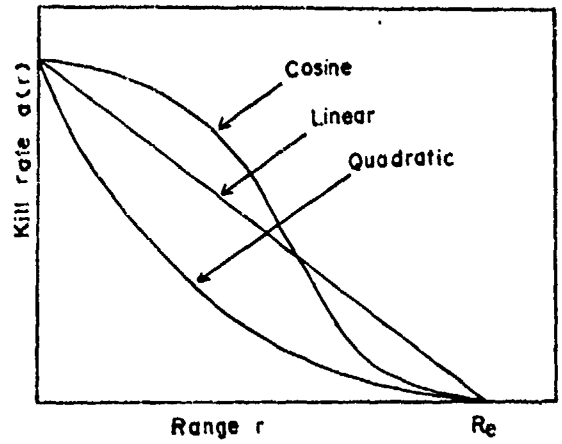
Three forms of kill rate were considered by Bonder. They are:

Linear form: $a_\ell(r) = k_a(1 - \frac{r}{R_e})$ for $r \leq R_e$

$= 0$ for $r > R_e$

Quadratic form: $a_q(r) = k_a(1 - \frac{r}{R_e})^2$

Cosine form: $$a_c(r) = \frac{k_a}{2}\left[1 + \cos\left(\frac{\pi r}{R_e}\right)\right]$$



When v is constant and has a positive sign if the Blue force decreases the range between the forces and has a negative sign if the Blue force increased the range and when both weapon systems have same effective range Re with opening range of the engagement R_o, the solutions to the equations (5) and (6) with linear attrition-rate coefficient are:

$$x(r) = x_o \cosh \Theta(r) + \sqrt{\frac{k_a}{k_b}}\, y_o \sinh \Theta(r) \qquad (7)$$

$$y(r) = y_o \cosh \Theta(r) + \sqrt{\frac{k_b}{k_a}}\, x_o \sinh \Theta(r) \qquad (8)$$

where $$\Theta(r) = \frac{\sqrt{k_a k_b}}{2v}\left[(R_e - r)^2 - (R_e - R_o)^2\right]$$

With quadratic and cosine attrition rate coefficient, equations (7) and (8) still valid but the $\Theta(r)$ become respectively:

$$\theta(r) = \frac{\sqrt{k_a k_b}}{3vR_e^{\,2}} \left[(R_e - r)^3 - (R_e - R_o)^3\right]$$

$$\theta(r) = \sqrt{\frac{k_a k_b}{2v}}\,(R_o - r) + \frac{R_e\sqrt{k_a k_b}}{2\pi v}\left[\sin\left(\frac{\pi R_o}{R_e}\right) - \sin\left(\frac{\pi r}{R_e}\right)\right]$$

3. The Assumptions

The above Lanchester-type equations and the riverine fighting after the initial ambush are based upon the following assumptions:

a. Two opposing forces are engaged in a fight. Units on each side are identical but the rate of attrition caused to the opponent may be different for each force.

b. When no artillery is used and when neither side tries to break contact, each unit on either side is within effective weapon range of all units of the other side.

c. Each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target.

d. Fire is uniformly distributed over remaining units.

e. If the Blue force decides to turn around following course A or B (Fig. 5), it would choose to turn around at the range between forces and it would turn right away with no time lost.

f. The fight begins right after the initial ambush and this starting time also coincides with the moment when the Blue force starts to return fire.

g. The range between the two forces increases or decreases uniformly, i.e., the Blue force would increase or decrease its speed following the courses A or B or C in such a way that the relative speed of the Blue force with respect to the Red force is always constant.

h. The attrition rate coefficients of both forces have a linear form when the Blue force follows either course A or B or C.

i. When the Red force breaks contact, due to the terrain conditions onshore, the Blue force has to use the artillery while the Red force cannot return fire anymore.

j. When the Red force breaks contact, its force would be dispersed right away, therefore causing the Blue force to use area fire with its artillery. The Blue force artillery attrition rates become time dependent and vary exponentially with negative time while the Red force weapon attrition rates become zero. Thus, the Lanchester-type equations become in this case:

$$\frac{dx}{dt} = 0 \tag{8}$$

$$\frac{dy}{dt} = -b(t)xy = -k_b' e^{-t} xy \tag{9}$$

Therefore $x = x_o$

where x_o is a constant and equal to the number of tubes of artillery of the Blue force at the time the Red force breaks contact.

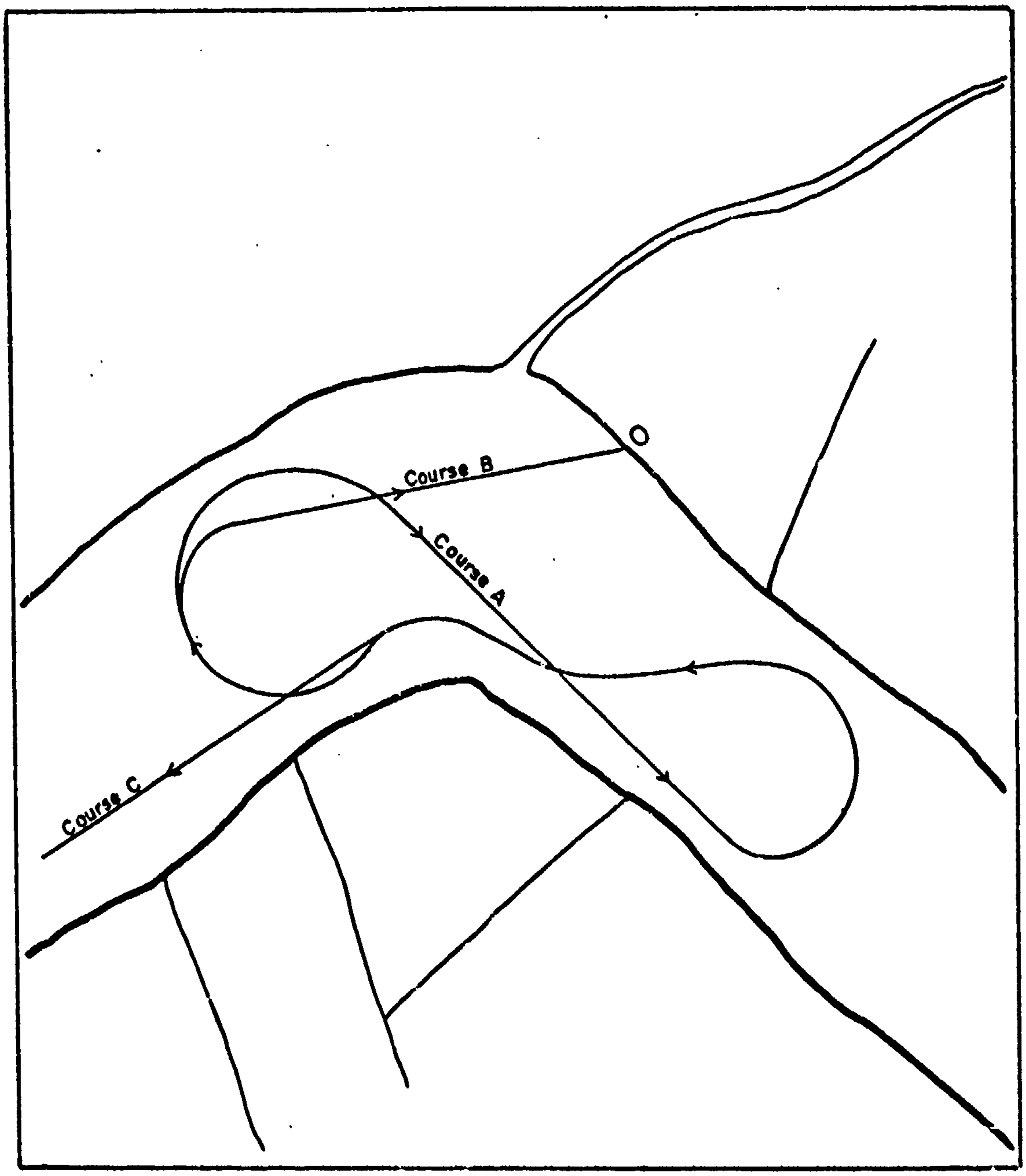


Figure 5. The Strategies After the Ambush.

Equation (9) becomes

$$\int_{y_o}^{y(t)} \frac{dy}{y} = -k_b' x_o \int_o^t e^{-t}\, dt$$

or $\qquad \log y(t) = \log y_o + k_b' x_o (e^{-t} - 1)$

Finally: $\qquad y(t) = y_o e^{k_b' x_o (e^{-t} - 1)}$

in which $\qquad y_o$ = initial number of survivors of the Red force at the time it breaks contact

and $\qquad k_b'$ = Blue force artillery attrition rate at the time the Red force breaks contact

k. When the Blue force breaks contact by following course C, it would increase its speed up to 1.5 v.

4. The Input Data

a. The Attrition Rates

As analysis of the model proceeds, it is determined that the values given in the literature for attrition rates are too high. A battle fought with either side having an attrition rate of 0.04 as used by Schaffer would be over in a couple of seconds. Further investigation of Schaffer's value reveals that he has used a small target approximation with a value of 0.1 ft^2 for the exposed area of a rifleman. This is an extremely small value and could not be justified by any analysis. An attempt is made to use Barfoot's harmonic mean [Ref. 15] of the Lanchester attrition rate coefficient:

$$k_a = \frac{1}{E[T]}$$

where T is time to destroy target

$$E[T] = t_a + t_1 + t_h + \frac{(t_a - t_f)}{P(K|H)} + \frac{(t_m - t_f)}{P(h|m)}\left[\frac{1-P(h|h)}{P(K|H)} + P(h|h) - p\right]$$

where [Ref. 16]

t_a = time to acquire target

t_1 = time to fire first round after target acquired

t_h = time to fire a round after sensing hit on previous round

t_m = time to fire a round after sensing miss on previous round

t_f = time of flight of projectile

$P(K|H)$ = probability of kill given a hit

A special case assumes that:

(1) target acquisition time negligible $t_a = 0$

(2) $t_1 = t_h = t_m = t = 1/v$ where v = rate of fire

(3) independent rounds $P(h|m) = P(h|h = p =$ single shot hit probability then

$$k_a = v\ P(K|H)p$$

$$p = \frac{a}{\sigma\sqrt{2\pi}}$$

a = length of target in one dimensional space

σ^2 = variance of impact point

Suppose that a = 0.5 ft, σ = 9 ft, v = 10 rounds/minute, $P(K|H) = 0.25$ then p = 0.02 and $k_a = (10)(0.25)(0.02) = 0.05$ therefore the values are still unrealistic.

In order to get around this problem, values of 2.10^{-5} and 2.10^{-6} are chosen for Blue force attrition rates coefficient for small arms (b_k) and the value of 0.02 is chosen for Blue force attrition rates coefficient for artillery.

b. The Casualties

Although H. K. Weiss (1953) has noted that one of the deficiencies in the original Lanchester theory is that engagements that continue until one side is wiped out are rare and retreat begins when the number of casualties approaches the order of 10%, in this problem of a riverine ambush, according to combat experiences in Vietnam, it is assumed that the Red force would break contact when n = 20% and the Blue force would break contact when m = 25%. Furthermore when the Blue force breaks contact, it would run away for good without using any artillery.

c. The Artillery

It was assumed that the patrol boat had g = 4 tubes of artillery with the effective range of 2000 meters. They are assumed to be intact through-out the fighting and to be used for a period of p = 20 minutes.

d. The Speed

v = 7 knots = 3.5 meters/second

5. <u>The Output Data</u>

Numerical results for the model are obtained from the three computer program written in FORTRAN IV language; this program is shown on pages 67, 68, 69, 70. These

programs give the Blue and Red force strengths for every ten meters of either increasing or decreasing force separation when both forces use small arms. They also give the Red force strength when Red force breaks contact or when Blue force starts to use artillery. The casualty ratios between Red force and Blue force were obtained from these terminal force strengths.

Eight different cases are considered. The results are summarized in Table 6.

6. Discussion of Results

Table 6 shows the effects of attrition rate coefficients, initial force strengths and the strategy used on the outcome of the engagement, namely, the casualty ratios between the Red force and the Blue force.

Within this model, it appears that the third strategy offers the best solution to the patrol boat's Captain. However, when the attrition rates have the values around 10^{-5} which are relatively high, there is not much difference between the outcomes in all three strategies. Therefore, it would be better for the boat to use the second strategy since in this strategy, the effects of psychology were not considered. These effects were accomplished when the ambushers suddenly saw something big and made with steel coming right in front of them, firing at them of such volume and accuracy that they would rather seek cover by running away than continue to fire their weapons.

Casualty Ratios CR							
Case I	Case II	Case III	Case IV	Case V	Case VI	Case VII	Case VIII
$k_a=1/2k_b=10^{-5}$ $x_o=y_o=20$	$k_a=1/2k_b=10^{-6}$ $x_o=y_o=20$	$k_a=1/2k_b=10^{-5}$ $x_o=3/2y_o=30$	$k_a=1/2k_b=10^{-6}$ $x_o=3/2y_o=30$	$k_a=2k_b=10^{-5}$ $x_o=y_o=20$	$k_a=2k_b=10^{-6}$ $x_o=y_o=20$	$k_a=2k_b=10^{-5}$ $x_o=3/2y_o=30$	$k_a=2k_b=10^{-6}$ $x_o=3/2y_o=30$
First Strategy							
CR=3.28	CR=3.36	CR=4.88	CR=5.19	CR=0.66	CR=0.67	CR=1.18	CR=1.21
Second Strategy							
CR=3.28	CR=3.51	CR=4.88	CR=5.20	CR=0.66	CR=1.22	CR=1.18	CR=1.48
Third Strategy							
CR=3.27	CR=6.20	CR=5.02	CR=7.21	CR=0.67	CR=2.6	CR=1.20	CR=2.85
Optimum Strategy							
Any	3rd Strategy	3rd Strategy	3rd Strategy	Any	3rd Strategy	Any	3rd Strategy

Table 6. Casualty Ratio.

When the attrition rates have the values around 10^{-6} which are normal then there is a difference between the three strategies. Although in this case the third strategy is again the optimum strategy, the second strategy is nevertheless worth-while to be seriously considered. There are three reasons for this consideration. First of all, it would boast the morale of the boat's crew as well as the morale of the unit to which the patrol boat belongs. Secondly, the casualty ratios are higher than the ones in the first strategy and thirdly, as mentioned above, the effects of psychology and suppression are no less important.

The Blue force would try to break contact in all strategies when case V is the current situation. However, since all three casualty ratios have the same value which is in favor of the Red force, the second strategy seems again to be the best solution to the patrol boat if the three above considerations are taken into account.

C. SECOND SOLUTION: A STOCHASTIC MODEL

In the probabilistic development of the Lanchester theory, some of the desired results are the following:

1. the probability of m, n survivors at time t
2. the probability that one side wins
3. the expected number of survivors

However, when the original force strengths M and N are of any realistic size, the solutions become too complicated to be of direct practical use [Ref. 10]. Therefore, this

model can be applied to the problem of a riverine ambush when both sides, the Red force and the Blue force, use artillery or rockets with the assumption that when either side can score a direct hit, it would put out of action x per cent of his opponent force.

1. The Strategies

After the initial ambush, if the ambushers on the Red force continue to use artillery to attack the patrol boat, and if the patrol boat's Captain judges that it would be better for his side to counter-attack with his atrillery, then he is usually assumed to have two options:

a. Slow down the boat and try to stay in one place so that the counter-attack can be launched with greatest accuracy. This return-fire would go on for a planning horizon of t minutes.

b. Turn left and try to break contact by going full straight ahead and at the same time, use artillery to fire on enemy position until both sides are out of their weapon systems effective range. In this case, due to terrain conditions and also due to types of weapon systems, it is assumed that the attrition rates do not vary with range and that they are small compared with the ones in the first strategy.

2. The Model

The basic assumptions for stochastic combat formulations are:

a. The attrition process is Markov. The Markov property assumes that, from any instant of time, the behavior of the system depends on the state of the system at that instant and not on the previous history of the system.

b. The process possesses a stationary transition mechanism. A stationary transition mechanism assumes that events which occur in a given time interval depend only on the state of the system at the beginning of the interval, and on the length of the interval - not on the instant at which the time interval begins.

c. During interval of length Δt, the probability of both forces simultaneously losing a unit is negligible and the probability of more than one loss on a side is negligible.

Let

$P(m,n,t)$ = probability that there are m,n survivors after a time interval (o,t)

$A(m,n)$ = Blue force weapon attrition rate which is equal to $k_a n$ for square-law attrition process

$B(m,n)$ = Red force weapon attrition rate which is equal to $k_b m$ for square-law attrition process

$A(m,n)\Delta t$ = probability of one Blue casualty in interval from t to $t+\Delta t$

$B(m,n)\Delta t$ = probability of one Red casualty in interval from t to $t+\Delta t$

Based on the above assumptions, the system can arrive at state (m,n) after the interval $(o,t+\Delta t)$ in three mutually exclusive ways

a. (m,n) survivors at time t, 0 Blue and 0 Red casualties in Δt

b. (m+1,n) survivors at time t, 1 Blue and 0 Red casualties in Δt

c. (n,n+1) survivors at time t, 0 Blue and 1 Red casualties in Δt

Observe that:

Probability of no casualty on either side during interval from t to $t+\Delta t = [1-A(m,n)\Delta t][1-B(m,n)\Delta t] = 1 - [A(m,n) + B(m,n)]\Delta t + 0((\Delta t)^2)$

Hence, by ignoring term $0((\Delta t)^2)$:

$$\begin{aligned} P(m,n,t+\Delta t) = {} & P(m,n,t)[1-A(m,n)\Delta t - B(m,n)\Delta t] \\ & + P(m+1,n,t)A(m+1,n)\Delta t \\ & + P(m, n+1, t)B(m, n+1)\Delta t \end{aligned}$$

Taking the limit of

$$\lim_{\Delta t \to 0} \frac{P(m,n,t+\Delta t) - P(m,n,t)}{\Delta t}$$

one obtains the Komogorov forward or pure death equation:

$$\begin{aligned} \frac{dP(m,n,t)}{dt} = {} & P(m+1,n,t)A(m+1,n)+P(m,n+1,t)B(m,n+1) \\ & -[A(m,n)+B(m,n)]P(m,n,t) \end{aligned}$$

With stochastic square-law attrition process, this equation becomes:

$$\frac{dP(m,n,t)}{dt} = anP(m+1,n,t)+bmP(m,n+1,t) - (bm+an)P(m,n,t)$$

To solve this equation for the case m = M-1, n = N:

$$\frac{dP(M-1,N,t)}{dt} = aNP(M,N,t)-[b(M-1)+aN]P(M-1,N,t)$$

with $P(M-1,N,0) = 0$

one may recall that:

$$\frac{dP(M,N,t)}{dt} = -(bM+aN)P(M,N,t)$$

with $P(M,N,0) = 1$

hence $P(M,N,t) = e^{-(bM+aN)t}$

Therefore, by letting $p(t) = P(M-1,N,t)$, one gets:

$$\frac{dp(t)}{dt} + [b(M-1)+aN]p(t) = aNe^{-(bM+aN)t}$$

Then, by making the left hand side of this differential equation exact and after integrating, one gets [Ref. 15]:

$$P(M-1,N,t) = \frac{aN}{b}(1-e^{-bt})\, e^{-[b(M-1)+aN]t}$$

With similar development, for the case m = M-2, n = N, one case also have

$$P(M-2,N,t) = \frac{1}{2}\left(\frac{aN}{b}\right)^2 (e^{bt}-1)^2\, e^{-[bM+aN]t}$$

3. The Input and Output Data

For a fixed planning horizon of t = 20 minutes, let the artillery attrition rates when the boat stays in one place in the first strategy be ten times greater the artillery attrition rates when the boat tries to break contact in the second strategy.

In order to obtain what could be considered reasonable and valid results, values of 0.01, 0.02, 0.04 and 0.08 are chosen for the artillery attrition rates when the boat stays in one place. These values are chosen from several which have been used in an analysis of the model on the computer.

To determine which strategy is optimu, it is decided to use the ratios between the probability P(M-1,N,t) and the probability P(M,N-1,t). The strategy that has lower ratio is considered as better than the other.

When M=N=4, P(M-1,N,t) represented the probability that there are 3 groups left in the 4 group bloc of the Blue force and the 4 group bloc of the Red force remains intact. Similar representation for P(M,N-1,t). In some cases where there is a fluctuation of the values of the probabilities, it is decided to take the value of the ratio in the last time incremental step.

Table 7 summarizes the input and output data of the model.

Probability Ratio PR					
Case I	Case II	Case III	Case IV	Case V	Case VI
$k_a=1/2k_b=0.01$ M=N=4	$k_a=1/2k_b=0.04$ M=N=4	$k_a=1/2k_b=0.01$ M=3/2N=6	$k_a=1/2k_b=0.04$ M=3/2N=6	$k_a=2k_b=0.02$ M=N=4	$k_a=2k_b=0.08$ M=N=4
First Strategy					
PR=0.55	PR=0.78	PR=0.37	PR=0.52	PR=1.81	PR=1.28
Second Strategy					
PR=0.50	PR=0.52	PR=0.34	PR=0.34	PR=1.98	PR=1.93
Optimum Strategy					
Any	2nd Strategy	Any	2nd Strategy	Any	1st Strategy

Table 7. The Probability Ratio.

4. Discussion of Results

Table 7 shows the effects of the artillery attrition rate coefficients, initial force strengths and the strategy used on the outcome of the engagement as measured by the ratio of elements of the probability state vector, namely

$$\frac{P(M-1,N,t)}{P(M,N-1,t)}$$

Computations were performed with many different attrition rates judgmentaly selected. However, in many cases, the combat outcomes that resulted were quite intuitively inadmissible in that they did not agree with the author's past combat experiences and judgments.

Within this model, when the attrition rates are relatively small which have the values around 0.01 (cases I, II and V), there seems to be no difference between the available strategies, regardless of the force strengths.

When the attrition rates are around 0.04 (cases II, IV and VI) if the Red force has higher attrition rates then it would be better for the Blue force to try to break contact by choosing the second strategy. If the Red force has lower attrition rates then the first strategy seems to become optimum for the Blue force.

D. THIRD SOLUTION: A STOCHASTIC DUEL WITH DISPLACEMENT

In this stochastic duel two contestants A and B begin with infinite ammunition supplies and fire simultaneously at intervals (which need not be specified). On each round

fired each contestent has a fixed probability, p_A and p_B, of killing his opponent. However, if he misses he may either miss completely with probability r_A and r_B or he may have a "near miss" with probability g_A and g_B. In this event of a "near miss" the opponent must displace and take up a new firing position. It is assumed that in making a displacement a contestant loses one firing time. That is, the time to displace is the same as the time between rounds [Ref. 17].

Thus, it can be seen that this model can be applied to the problem of a riverine ambush when both sides, the Blue force and the Red force, form as two separate blocs which are the two contestants A and B described above. The force which has greater probability in scoring a hit on its target is declared the winner. It is also assumed that with this type of duel, both forces use artillery and that the Red force cannot score a "near miss" since as it can be seen in the following strategies, the Blue force is always in the move whether it gets a miss or a "near miss". Furthermore, the crew onboard the patrol boat do not have to displace when the Red force can score a "near miss".

1. The Strategies

The strategies of the Blue force after the initial ambush in this case are similar to the ones in the second solution, namely:

a. Slow down the boat and do not try to stay immobile but try to move in a haphazard way so that its

probability of being hit would be less although this may also reduce its fire-power accuracy.

b. Turn left and try to break contact by going straight ahead at full speed and at the same time use artillery to fire on enemy position until both sides are out of their weapon systems effective range. The probabilities do not vary with range and the probabilities of being hit are small compared with the ones in the first strategy.

2. <u>The Model</u>

Let 0 represent the state that a force bloc is in a firing condition (this means it did not receive a near miss on the previous round) and let x represent the state that a force bloc is in a displacement condition and cannot fire (that means that he received a near miss on the previous round). State probabilities for the pair A,B on any given round n may now be represented by the notayion $f_n(..)$ where the first position in the argument represents the state of A or the Blue force and the second of B or Red force. Thus four states for the pair are possible on any given round, n.

$f_n(00)$ = probability that A, B fire on n^{th} round

$f_n(0x)$ = probability that A fires and B does not on the n^{th} round

$f_n(x0)$ = probability that B fires and A does not on the n^{th} round

$f_n(xn)$ = probability that neither fires on the n^{th} round.

The matrix which gives the transition probabilities from any state-pair to any other state-pari is

from \ to	00	0x	x0	xx
00	$r_A r_B$	$g_A r_B$	$g_B r_A$	$g_A g_B$
0x	r_A	g_A	0	0
x0	r_B	0	g_B	0
xx	1	0	0	0

Thus the following difference equations may be written:

$$f_n(00) = f_{n-1}(00) r_A r_B + f_{n-1}(0x) r_A + f_{n-1}(x0) r_B + f_{n-1}(xx)$$

$$f_n(0x) = f_{n-1}(00) g_A r_B + f_{n-1}(0x) g_A$$

$$f_n(x0) = f_{n-1}(00) g_B r_A + f_{n-1}(x0) g_B$$

$$f_n(xx) = f_{n-1}(00) g_A g_B$$

where $p_A + r_A + g_a = p_B + r_B + g_B = 1.$

The initial conditions are:

$$f_0(00) = 1, \quad f_0(0x) = f_0(x0) = f_0(xx) = 0$$

The probability that A or Blue force wins is

$$P(A) = p_A(1-p_B) \Sigma_{n=o}^{n=\infty} f_n(00) + p_A \Sigma_{n=o}^{n=\infty} f_n(0x)$$

Ancker, Jr. and Williams [17] by transforming the recurrence formulas into algebraic relations, gave the solutions to P(A) as follows:

$$P(A) = \frac{X_A (R - X_B)}{R(X_A + X_B - X_A X_B)}$$

where

$$X_A = \frac{p_A}{1-g_A} = \frac{p_A}{p_A + r_A}$$

$$X_B = \frac{p_B}{1-g_B} = \frac{p_B}{p_B+r_B}$$

$$R = \frac{1-g_A g_B}{1-g_B}$$

Similar formula is obtained for P(B).

In the case of a riverine ambush, $g_B = 0$ or $R = 1$ and $X_B = p_B$

$$P(A) = \frac{X_A(1-p_B)}{X_A - X_A p_B + p_B}$$

where

$$X_A = \frac{p_A}{p_A+r_A}$$

$$P(B) = \frac{p_B(R'-X_A)}{R'\ (p_B+X_A-p_B X_A)}$$

where

$$R' = \frac{1}{1-g_A}$$

3. <u>The Input and Output Data</u>

Suppose $p_A = 0.2 \quad p_B = 0.3$

$r_A = 0.6 \quad r_B = 0.7$

$g_A = 0.2 \quad g_B = 0$

then

$X_A = 0.25$, $X_B = 0.3$, $R_1 = 1.25$

$P(A) = 0.369 \quad P(B) = 0.505$

Thus the Blue force has smaller probability of winning, therefore it would better use the second strategy, i.e., try to break contact with the Red force.

4. <u>Discussion of Results</u>

The values of P(A) and P(B) thus obtained depend on the different values of X_A, X_B and R which in turn depend on the values of p_A, p_B, r_A, r_B, g_A, g_B.

P(A) and P(B) can be plotted as contour map in Fig. 6. With any set of values p_A, p_B, r_A, r_B, g_A, g_B, whenever P(A) is greater than P(B), for the problem of a riverine ambush, the first strategy would be the optimum strategy for the patrol boat.

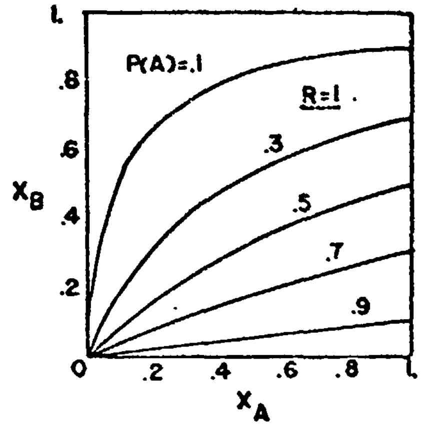


Figure 6.

V. CONCLUSIONS

The initial purpose of this thesis was to try to develop insight into an optimal strategy for the patrol boat patrolling along a river before being ambushed at a bend of the river and then an optimal strategy for the same patrol boat after the ambush had been initiated.

For modelling purposes, the actual (real-world) combat situation was conceptualized as consisting of several parts. In the first of these, a game-theoretic model and a statistical decision theory model were used for the problem before the ambush. It was found that the strategy to sail the boat close to the inside of the bend of the river seemed to be the optimum strategy for the Blue force in this stage of the game.

In the second stage of the game, i.e., after the ambush had been initated, three mathematical models of combat were used to try to find an optimal strategy for the patrol boat. The first model was deterministic. The second and third models were stochastic. In the stochastic model, the attrition rates were assumed to be constant.

To be able to actually compare the two models, deterministic versus probabilistic, it was necessary to have time or range-dependent attrition rates in both cases and to use the same weapons system. For this comparison, Bonder [Ref. 10] notes that H. Weiss has observed that there exists very little difference between probabilistic

flow of the solution and deterministic results for forces involving more than a few dozen men. Taylor [Ref. 18] has also stated that a deterministic attrition model and the expected value of the corresponding stochastic attrition model should be in close agreement for large numbers of combattants and small time. Otherwise there would be less survivors for deterministic model than for stochastic analogue.

```
C        PROGRAM TO DETERMINE THE EFFECTS OF ATTRITION
C        RATE COEFFICIENTS AND FORCE STRENGTH UPON THE
C        TERMINAL FORCE STRENGTHS WHEN THE RANGE BETWEEN
C        FORCES VARIES CONTINUOUSLY.
C
C
C         A DYNAMIC COMBAT MODEL WITH DETERMINISTIC
C        APPROACH APPLIED TO THE PROBLEM OF A RIVERINE
C        AMBUSH.
C
C
C
C        EXPLANATION OF VARIABLE NAMES:
C
C        AK   :  RED FORCE SMALL ARMS ATTRITION RATES
C                AT THE OPENING RANGE
C        BK   :  BLUE FORCE SMALL ARMS ATTRITION RATES AT
C                THE OPENING RANGE
C        V    :  RELATIVE SPEED OF BLUE FORCE WITH RESPECT
C                TO RED FORCE
C        DI   :  II : III : INITIAL BLUE FORCE STRENGTH
C        DJ   :  JJ : JJJ : INITIAL RED FORCE STRENGTH
C        KE   :  SMALL ARMS EFFECTIVE RANGE
C        KO   :  SMALL ARMS OPENING RANGE.
C        IX   :  PER CENT IN CASUALTY OF RED FORCE.
C        IP   :  NUMBER OF MINUTES DURING WHICH BLUE FORCE
C                USES ARTILLERY
C        BBK  :  BLUE FORCE ARTILLERY ATTRITION RATES
C        XX   :  NUMBER OF TUBES OF ARTILLERY OF BLUE FORCE
C        CR   :  CASUALTY RATIO BETWEEN RED FORCE
C                AND BLUE FORCE
C
C
C
C        FIRST STRATEGY
C
C
         DIMENSION AK(8),BK(8),V(8),DI(8),DJ(8),KF(8),
        1KO(8),JJJ(8),III(8),IX(8),IP(8),BBK(8),XX(8)
         DO 4000 L=1,8
         READ(5,100) AK(L),BK(L),V(L),DI(L),DJ(L),KF(L),
        1KO(L),JJJ(L),IX(L),IP(L),BBK(L),XX(L),III(L)
  100    FORMAT(3F8.6,2F4.2,5I4,2F8.6,I4)
  150    WRITE(6,200) AK(L),BK(L),V(L),DI(L),DJ(L),KF(L),
        1KO(L),IX(L),IP(L),BBK(L),XX(L)
  200    FORMAT('1',//20X,'INITIAL VALUES'///,18X,'AK=',
        1F8.6,2X,'BK=',F8.6,2X,'V=',F4.2,2X,'DI=',F4.1,2X
        2,'DJ=',F4.1//,18X,'KF=',I5,2X,'KO=',I4,2X,'IX=',
        3I3,2X,'IP=',I3,2X,'BBK=',F5.3,2X,'XX=',F4.2/////)
C
C
C        THIS PORTION CALCULATES FORCES TERMINAL
C        STRENGTH,I.E.,BLUE FORCE TERMINAL STRENGTH
C        AND RED FORCE TERMINAL STRENGTH
C
C
  250    M=KO(L)
         N=KF(L)
         DO 500 I=M,N,10
         TT=((SQRT(AK(L)*BK(L)))/(2*V(L)))*((KF(L)-I)**2
        1-(KF(L)-KO(L))**2)
         X=DI(L)*COSH(TT)+(DJ(L)*SQRT(AK(L)/BK(L)))*SINH(TT)
         Y=DJ(L)*COSH(TT)+(DI(L)*SQRT(BK(L)/AK(L)))*SINH(TT)
  290    WRITE(6,300) I,X,Y
  300    FORMAT(17X,I4,2F10.2)
C
C
C        RED FORCE BREAKS CONTACT WHEN SUFFERING A
C        CASUALTY OF 10IX PERCENT OF ITS INITIAL FORCE
C        STRENGTH.
C
C
```

```
      IF(Y.LE.IX(L)*JJJ(L)/10) GO TO 1410
      IF(I.GE.2*KO(L)) GO TO 600
      IF((X.LE.0.0).OR.(Y.LE.0.0)) STOP
  500 CONTINUE
C
C
C     BLUE FORCE TURNS AROUND WHEN THE RANGE BETWEEN
C     FORCES REACHES 400 METERS MARK
C
C
  600 V(L)=-V(L)
      X1=X
      Y1=Y
      M=KO(L)
      DO 1000 IT=1,M,10
      IR=2*KO(L)-IT-9
      TT1=((SQRT(AK(L)*BK(L)))/(2*V(L)))*((KE(L)-IR)
     1**2-(KE(L)-2*KO(L))**2)
      X=X1*COSH(TT1)+(Y1*SQRT(AK(L)/BK(L)))*SINH(TT1)
      Y=Y1*COSH(TT1)+(X1*SQRT(BK(L)/AK(L)))*SINH(TT1)
  800 WRITE(6,900) IR,X,Y
  900 FORMAT(17X,I4,2F10.2)
      IF(Y.LE.IX(L)*JJJ(L)/10) GO TO 1410
      IF(IR.GT.KO(L)) GO TO 910
C
C
C     BLUE FORCE CONTINUES TO TURN AROUND IF THE
C     SITUATION PERMITS
C
C
      DI(L)=X
      DJ(L)=Y
      V(L)=-V(L)
      GO TO 250
  910 IF(X.LE.0.0) STOP
 1000 CONTINUE
C
C
C     BLUE FORCE STARTS TO USE ARTILLERY
C
C
 1410 MM=IP(L)
      DO 1460 JK=1,MM,4
      ZT=-JK
      YZ=Y*EXP(BBK(L)*XX(L)*(EXP(ZT)-1.0))
      CR=(JJJ(L)-YZ)/(III(L)-X)
      WRITE(6,1430) JK,YZ,CR
 1430 FORMAT(17X,I4,10X,2F10.2///)
 1460 CONTINUE
 4000 CONTINUE
      STOP
      END



C     SECOND STRATEGY
C
C
      DIMENSION AK(8),BK(8),V(8),II(8),JJ(8),KE(8),
     1KO(8),JJJ(8),III(8),IX(8),IP(8),BBK(8),XX(8)
      DO 4000 L=1,8
      READ(5,100) AK(L),BK(L),V(L),II(L),JJ(L),KE(L),
     1KO(L),JJJ(L),IX(L),IP(L),BBK(L),XX(L),III(L)
  100 FORMAT(3F8.6,7I4,2F8.6,I4)
  150 WRITE(6,200) AK(L),BK(L),V(L),II(L),JJ(L),KE(L),
     1KO(L),IX(L),IP(L),BBK(L),XX(L)
  200 FORMAT('1',//20X,'INITIAL VALUES'///,18X,'AK=',
     1F8.6,2X,'BK=',F8.6,2X,'V=',F4.2,2X,'II=',I4,2X
     2,'JJ=',I4//,18X,'KE=',I5,2X,'KO=',I4,2X,'IX=',
     3I3,2X,'IP=',I3,2X,'BBK=',F5.3,2X,'XX=',F4.2/////)
      M=KO(L)
```

```
      N=KE(L)
  250 DO 500 I=M,N,10
      TT=((SQRT(AK(L)*BK(L)))/(2*V(L)))*((KE(L)-I)**2
     1-(KE(L)-KO(L))**2)
      X=II(L)*COSH(TT)+(JJ(L)*SQRT(AK(L)/BK(L)))*SINH(TT)
      Y=JJ(L)*COSH(TT)+(II(L)*SQRT(BK(L)/AK(L)))*SINH(TT)
  280 WRITE(6,300) I,X,Y
  300 FORMAT(17X,I4,2F10.2)
C
C
C     RED FORCE BREAKS CONTACT WHEN SUFFERING A
C     CASUALTY OF 10IX PERCENT OF ITS INITIAL FORCE
C     STRENGTH.
C
C
      IF(Y.LE.IX(L)*JJJ(L)/10) GO TO 1410
      IF(I.GE.2*KO(L)) GO TO 600
      IF((X.LE.0.0).OR.(Y.LE.0.0)) STOP
  500 CONTINUE
C
C
C     BLUE FORCE TURNS AROUND WHEN THE RANGE BETWEEN
C     FORCES REACHS 400 METERS MARK AND GOES STRAIGHT
C     TO THE ENEMY'S POSITION.
C
C
  600 V(L)=-V(L)
      X1=X
      Y1=Y
      NN=2*KO(L)
      DO 1000 IT=1,NN,10
      IR=2*KO(L)-IT
      TT1=((SQRT(AK(L)*BK(L)))/(2*V(L)))*((KE(L)-IR)
     1**2-(KE(L)-2*KO(L))**2)
      X=X1*COSH(TT1)+(Y1*SQRT(AK(L)/BK(L)))*SINH(TT1)
      Y=Y1*COSH(TT1)+(X1*SQRT(BK(L)/AK(L)))*SINH(TT1)
  800 WRITE(6,900) IR,X,Y
  900 FORMAT(17X,I4,2F10.2)
      IF(Y.LE.IX(L)*JJJ(L)/10) GO TO 1410
 1000 CONTINUE
C
C
C     BLUE FORCE STARTS TO USE ARTILLERY
C
C
 1410 MM=IR(L)
      DO 1450 JK=1,MM,4
      ZT=-JK
      YZ=Y*EXP(BRK(L)*XX(L)*(EXP(ZT)-1.0))
      CR=(JJJ(L)-YZ)/(III(L)-X)
      WRITE(6,1430) JK,YZ,CR
 1430 FORMAT(17X,I4,10X,2F10.2///)
 1450 CONTINUE
 4000 CONTINUE
      STOP
      END


C     THIRD STRATEGY
C
C
      DIMENSION AK(8),BK(8),V(8),II(8),JJ(8),KE(8),
     1KO(8),JJJ(8),III(8),IX(8),IR(8),BRK(8),XX(8)
      DO 4000 L=1,8
      READ(5,100) AK(L),BK(L),V(L),II(L),JJ(L),KE(L),
     1KO(L),JJJ(L),IX(L),IR(L),BRK(L),XX(L),III(L)
  100 FORMAT(3F8.6,7I4,2F8.5,I4)
  150 WRITE(6,200) AK(L),BK(L),V(L),II(L),JJ(L),KE(L),
     1KO(L),IX(L),IR(L),BRK(L),XX(L)
  200 FORMAT('1',//20X,'INITIAL VALUES'///,18X,'AK=',
```

```
     1F8.6,2X,'BK=',F8.6,2X,'V=',F4.2,2X,'II=',I4,2X
     2,'JJ=',I4//,18X,'KE=',I5,2X,'KO=',I4,2X,'IX=',
     3I3,2X,'IP=',I3,2X,'BBK=',F5.3,2X,'XX=',F4.2/////)
C
C
C      THIS PORTION CALCULATES FORCES TERMINAL
C      STRENGTH,I.E.,BLUE FORCE TERMINAL STRENGTH
C      AND RED FORCE TERMINAL STRENGTH
C
C      BLUE FORCE TURNS LEFT AND TRIES TO BREAK CONTACT
C      BY GOING STRAIGHT AHEAD AT FULL SPEED.
C
C
      M=KO(L)
      N=KE(L)
  250 DO 500 I=M,N,10
      TT=((SQRT(AK(L)*BK(L)))/(2*V(L)))*((KE(L)-I)**2
     1-(KE(L)-KO(L))**2)
      X=II(L)*COSH(TT)+(JJ(L)*SQRT(AK(L)/BK(L)))*SINH(TT)
      Y=JJ(L)*COSH(TT)+(II(L)*SQRT(BK(L)/AK(L)))*SINH(TT)
  280 WRITE(6,300) I,X,Y
  300 FORMAT(17X,I4,2F10.2)
C
C
C      RED FORCE BREAKS CONTACT WHEN SUFFERING A
C      CASUALTY OF 10IX PERCENT OF ITS INITIAL FORCE
C      STRENGTH.
C
C
      IF(Y.LE.IX(L)*JJJ(L)/10) GO TO 1410
  500 CONTINUE
C
C
C      BLUE FORCE STARTS TO USE ARTILLERY
C
C
 1410 MM=IP(L)
      DO 1460 JK=1,MM,2
      ZT=-JK
      YZ=Y*EXP(BBK(L)*XX(L)*(EXP(ZT)-1.0))
      CR=(JJJ(L)-YZ)/(III(L)-X)
      WRITE(6,1430) JK,YZ,CR
 1430 FORMAT(17X,I4,10X,2F10.2///)
 1460 CONTINUE
 4000 CONTINUE
      STOP
      END
```

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